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A PLANE PROBLEM

IN FLUID FLOW

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "A PLANE PROBLEM IN FLUID FLOW" submitted by BERNARD TULLY REILLY in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

A theoretical analysis of the plane flow produced by the relative normal motion of two parallel plates is presented. Creeping motion of a Newtonian liquid is considered so that a linear system results. The classical solution is presented and shown to lead to rather artificial boundary conditions. These boundary conditions are modified by the superposition of an auxiliary solution which is obtained by expanding the stream function in a double Fourier series. For large plate width to separation ratio the auxiliary solution is found only to affect the flow in the region of the plate edges.

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NOMENCLATURE

μ	coefficient of viscosity
λ	coefficient of viscosity
ρ	density
σ_{ij}	stress tensor referred to rectangular coordinate axes ox_i
s_{ij}	stress deviator tensor
p	thermodynamic pressure
u_i	velocity vector
e_{ij}	$= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ rate of deformation tensor
x, y	rectangular cartesian coordinates
u, v	velocity components in the x and y directions respectively
X, Y	nondimensional rectangular cartesian coordinates
U, V	nondimensional velocity components
ψ	stream function
Ψ	nondimensional stream function

CHAPTER I

INTRODUCTION

This thesis considers isothermal, creeping, laminar flow of an incompressible Newtonian fluid. Although many applications of fluid mechanics, such as hydrodynamic lubrication, involve heat dissipation, the assumption of isothermal flow eliminates the complication of variable viscosity due to temperature gradients and does not impose a great restriction on the use of the analysis. Creeping flow is considered and this results in linear differential equations. However one of the objects of this thesis is to indicate a scheme for the inclusion of inertia effects.

With the governing equations linear, the superposition principle, which is often used in classical elasticity problems, can be applied.

Since liquids show negligible compressibility effects, even under extreme pressure, an incompressible fluid with ideal Newtonian properties is the mathematical model upon which the analysis of this thesis is based. Many similar problems have been solved subject to these restrictions and the results found to be in good agreement with experimental observation.

The classical solution given by Michell⁽⁸⁾ for creeping laminar flow between parallel plates with relative normal motion implies a parabolic distribution of the radial velocity component. This solution satisfies the Navier Stokes equation and the velocity boundary condition at the plates but gives a rather artificial stress distribution at the edges.

For large radius to separation ratios a principle analogous to St. Venant's principle in classical elasticity can be applied and these boundary effects neglected. A more satisfactory solution can be obtained by superimposing a solution on the parabolic profile and thereby satisfying the natural boundary condition of constant normal stress and zero shear stress at the edges.

To avoid the complexity of cylindrical coordinates the plane problem is considered. The origin of the coordinate system and the symmetrical movement of the plates were chosen to facilitate a possible later study of inertia effects. For the quasistatic problem the solution is not affected if a velocity of translation is superimposed so that one plate is stationary. With both plates moving the origin is taken at the centre in order to keep the frame of reference stationary.

The plane problem is solved using a Fourier series expansion of the stream function. The governing equations

necessitate differentiating this series four times. In two papers on the stability of viscous flow Goldstein^(3,4) shows how Fourier series can be used, the justification for differentiation depending on the conditions satisfied by the function at the end points of the interval. Green⁽⁵⁾ shows how double Fourier series can be applied to boundary value problems that involve fourth order partial differential equations. The use of a double series avoids some of the algebraic difficulties of a single series.

In order to satisfy the boundary conditions of the problem directly, the expansion of the normal velocity component at the plates in a single series is required so that it may be incorporated in the solution. A few terms of the resulting infinite series shows the Gibbs Phenomenon which is due to non uniform convergence at $x = \pm a$. Even after many terms are used the velocity profiles do not show conservation of volume and an overshoot develops at one point on the profile. The development is due to the Gibbs Phenomenon, which arises in the expansion of discontinuous functions.

Specified stress boundary conditions at $x = \pm a$ can be satisfied by superimposing an auxiliary solution on the parabolic solution. The auxiliary solution must satisfy the boundary conditions of zero velocity at $y = \pm b$.

CHAPTER II

GOVERNING EQUATIONS OF VISCOUS FLOW

The governing equations for laminar flow of an incompressible Newtonian viscous fluid are the continuity equation

$$\nabla \cdot \bar{U} = 0 \quad (2.1)$$

where $\bar{U}$ is the velocity and the Navier Stokes equation

$$\rho \frac{d\bar{U}}{dt} = -\nabla p + \rho \bar{X} + \mu \nabla^2 \bar{U} \quad (2.2)$$

where $\bar{X}$ is the body force per unit mass.

These equations applied to the two-dimensional finite region bounded by two parallel plates (Fig. 2.1) in the absence of body forces are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.2b)$$

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (2.3)$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

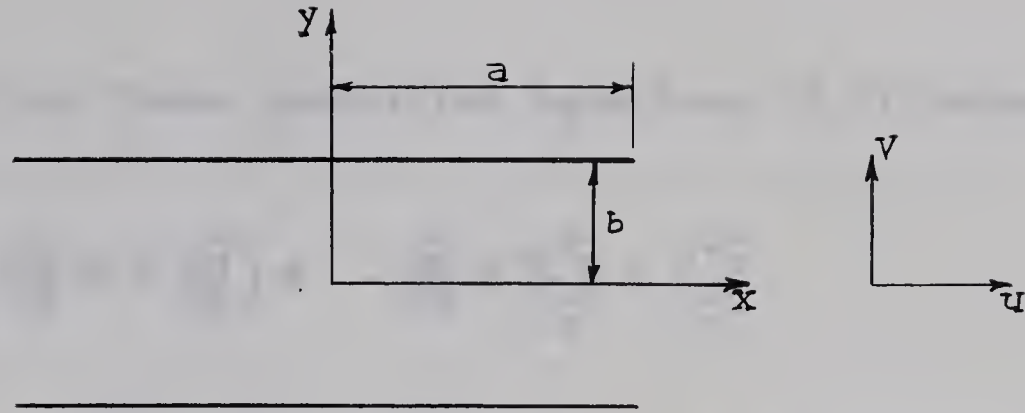


Fig. 2.1

The equations (2.3) can be reduced to non-dimensional form with the use of a suitable characteristic velocity and length for the problem. If the plates are allowed to separate with constant relative velocity W , their width being $2a$, the characteristic velocity and length become W and a .

The non-dimensional variables become

$$\bar{t} = \frac{tW}{a}, \quad \bar{x} = \frac{x}{a}, \quad \bar{y} = \frac{y}{a}$$

$$\bar{u} = \frac{u}{W}, \quad \bar{v} = \frac{v}{W}, \quad \bar{p} = \frac{ap}{\mu W}$$

Since a and b are considered in this problem to be of the same order all of the non-dimensional variables are $O(1)$.

Substituting these quantities equations (2.3) become

$$\frac{Wap}{\mu} \left(\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} \right) = - \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \quad (2.5)$$

$$\frac{Wap}{\mu} \left(\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} \right) = - \frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}$$

where $\frac{Wap}{\mu} = R$ the Reynolds number.

For creeping motion the Reynolds number approaches zero and since the terms in brackets are $O(1)$ the material derivative can be neglected. This is a form of Newtonian flow in which the pressure and viscous force are of the same order and the inertia term is neglected.

The approximation involved by neglecting the inertia term can be justified if the solution obtained by making the approximation satisfies the approximation used in obtaining it. This will be investigated in the concluding chapter and the simple solution obtained shown to satisfy the approximation.

For creeping motion where the material derivative is neglected equation (2.2) becomes

$$-\nabla p + \rho \bar{X} + \mu \nabla^2 \bar{U} = 0 \quad (2.8)$$

Considering only body forces which are derivable from a scalar potential ϕ equation (2.8) becomes

$$\nabla(p - \rho\phi) = \mu \nabla^2 \bar{U} \quad (2.9)$$

and for plane flow

$$\frac{\partial}{\partial x} (p - \rho\phi) = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (2.10)$$

$$\frac{\partial}{\partial y} (p - \rho\phi) = \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) .$$

From equation (2.9)

$$\text{curl } \nabla^2 \bar{U} = 0$$

or

$$-\frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 v}{\partial x \partial y^2} = 0 \quad (2.11)$$

The continuity equation implies the existence of a stream function ψ such that

$$u = \frac{\partial \psi}{\partial y} , \quad v = - \frac{\partial \psi}{\partial x} . \quad (2.12)$$

Substituting equations (2.12) in (2.11) gives

$$\frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 , \quad (2.13)$$

or
$$\nabla^4 \psi = 0 .$$

This is the differential equation for the stream function for creeping flow of an incompressible Newtonian fluid.

CHAPTER III

USE OF FOURIER SERIES

The method used in this thesis involves the differentiation of a Fourier series four times. In many applications of Fourier series the justification for differentiation is often overlooked. In two papers on the stability of viscous flow Goldstein^(3,4) showed how Fourier series can be used, the justification for differentiating depending on the conditions satisfied by the function at the end points of the interval. Green⁽⁵⁾ extended Goldstein's work to double Fourier series.

If $F(x,y)$ is defined and bounded in the range $(-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)$ and if $F(x,y)$ satisfies the conditions:

(a) $F(x,y)$ has only a finite number of maxima and minima
(b) $F(x,y)$ has only a finite number of finite discontinuities,
it may be expanded as a double Fourier series. These conditions are sufficient and are valid for the functions which arise in this application to fluid flow.

If $F(x,y)$ satisfies the Dirichlet conditions and if, outside the range $(-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)$ $F(x,y)$ is periodic in the sense that

$$F(x + 2\pi, y) = F(x, y)$$

$$F(x, y + 2\pi) = F(x, y)$$

then F can be represented by a double trigonometric series. For example a function $F(x,y)$ that is an odd function of x and y and has half range symmetry in x and y can be represented by

$$F(x,y) = \sum_{r=1,3,\dots}^{\infty} \sum_{s=1,3,\dots}^{\infty} A_{rs} \sin rx \sin sy$$

where the A_{rs} are defined by

$$A_{rs} = \frac{16}{\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} F(x,y) \sin rx \sin sy \, dx \, dy$$

For a fixed y the series converges to the sum

$$\frac{1}{2} \left[F(x+0, y) + F(x-0, y) \right]$$

where $F(x+0, y)$ is the value of F as x is approached from the larger value and $F(x-0, y)$ is the value of F as x is approached from the smaller value. Similarly for a fixed x the series converges to

$$\frac{1}{2} \left[F(x, y+0) + F(x, y-0) \right]$$

If $F(x,y)$ is continuous at the point this reduces to $F(x,y)$. Also due to the symmetry of the problem only odd values of r and s will appear and will not be shown explicitly.

If $F(x,y)$ is a function which can be expanded in a double Fourier sine-sine series in the region $(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2})$, and if its partial derivative $\partial F / \partial x$ can be expanded in a cosine-sine series, the coefficients in each

case being formed by the usual rule, then the coefficients in the second series can be related to those in the first series by a partial integration. Similar remarks are valid for sine-cosine, cosine-cosine, and cosine-sine series.

Thus if

$$F(x,y) = \sum \sum A_{rs} \sin rx \sin sy \left(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \right)$$

then

$$\frac{\partial F(x,y)}{\partial x} = \sum \sum r A_{rs} \cos rx \sin sy \left(-\frac{\pi}{2} < x < \frac{\pi}{2}, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \right)$$

Similarly if

$$F(x,y) = \sum \sum A_{rs} \cos rx \sin sy \left(-\frac{\pi}{2} < x < \frac{\pi}{2}, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \right)$$

and

$$\frac{\partial F(x,y)}{\partial x} = \sum \sum B_{rs} \sin rx \sin sy \left(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \right)$$

The B_{rs} are found by the calculation

$$\begin{aligned} B_{rs} &= \frac{16}{\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\partial F(x,y)}{\partial x} \sin rx \sin sy \, dx \, dy \\ &= \frac{16}{\pi^2} \int_0^{\frac{\pi}{2}} \left(F \sin rx \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} F r \cos rx \, dx \right) \sin sy \, dy \\ &= -r A_{rs} + \frac{16}{\pi^2} \int_0^{\frac{\pi}{2}} F\left(\frac{\pi}{2}, y\right) (-1)^{\frac{r-1}{2}} \sin sy \, dy \end{aligned}$$

Thus

$$\frac{\partial F(x,y)}{\partial x} = \sum \sum \left(r A_{rs} + (-1)^{\frac{r-1}{2}} d_s \right) \sin rs \sin sy$$

$$\left(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \right)$$

where $d_s = \frac{16}{\pi^2} \int_0^{\frac{\pi}{2}} F\left(\frac{\pi}{2}, y\right) \sin sy dy.$

The coefficients in the double Fourier series expansions of $F(x,y)$ and $\partial F(x,y)/\partial y$ can be found similarly.

CHAPTER IV

METHOD OF SOLUTION

4.1 Preliminary

The classical solution given by Michell for creeping laminar flow between parallel plates with relative normal motion is not completely satisfactory. Although the velocity field satisfies the Navier Stokes equations it results in a rather artificial stress condition at the edges. Incompressible flow is considered since this is a realistic assumption for a liquid. Inertia forces are neglected but body forces, obtained as the gradient of a scalar potential function can be incorporated.

The problem considered is the modification of the plane flow analogue of Michell's solution by the superposition of an auxiliary problem in order to satisfy the boundary condition of constant normal stress and zero shear stress at the edges.

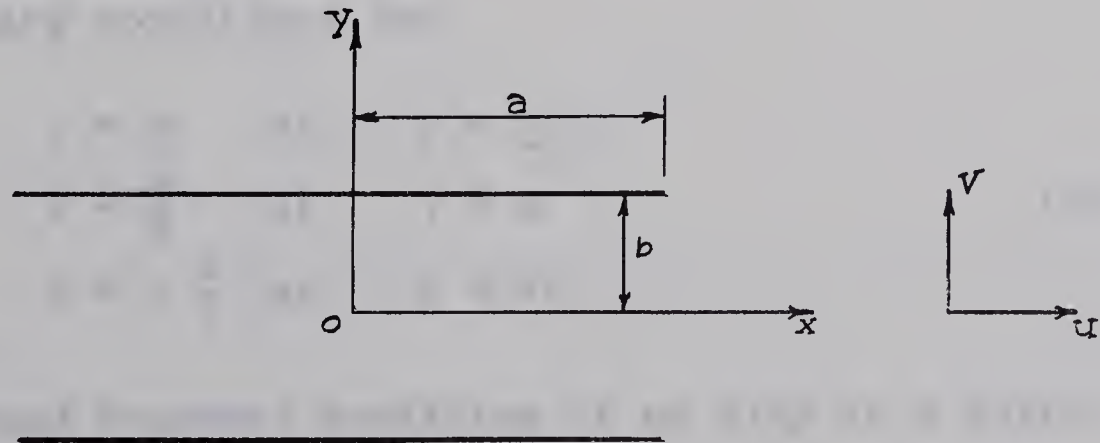


FIG. 4.1 COORDINATE SYSTEM

The origin 0 of the coordinate system is taken as shown in figure 4.1 to maintain a stationary coordinate system. The plates move with symmetrical normal motion, relative to the coordinate system, either apart or together. For the purposes of the analysis they are considered to move apart but the reverse effect involves only a trivial change of sign. Although the quasistatic case is not affected, except for a superimposed velocity of translation, by moving only one plate, the symmetrical motion is desirable for possible future consideration of inertia effects with the separation velocity a function of time.

The governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.2b)$$

$$\frac{\partial}{\partial x} (p - \rho\phi) = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (2.10)$$

$$\frac{\partial}{\partial y} (p - \rho\phi) = \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

and the boundary conditions are

$$\begin{aligned} u &= 0 & \text{at} & \quad y = \pm b \\ v &= \frac{W}{2} & \text{at} & \quad y = b \\ v &= -\frac{W}{2} & \text{at} & \quad y = -b. \end{aligned} \quad (4.1 \text{ a,b,c})$$

The imposed boundary condition of no slip at a solid boundary has been a subject of much discussion. Under most

circumstances this condition has been experimentally verified although it cannot be proved from hydrodynamical considerations. This boundary condition may not be realistic for extreme conditions of rarefied gases and very high velocity where molecular effects, which are not included in continuum theory, are significant. In hydrodynamical lubrication the no slip boundary condition is assumed to hold except where the film thickness becomes comparable with the mean free path of the molecules. This microscopic effect is also excluded from consideration.

4.2 Elementary Solution

The no slip boundary conditions (4.1a) are satisfied by the expression

$$u = cx(y^2 - b^2) \quad (4.2)$$

Substitution of (4.2) in the continuity equation (2.2a) gives

$$c(y^2 - b^2) + \frac{\partial v}{\partial y} = 0.$$

Consequently

$$v = -c \frac{y^3}{3} + cb^2 y + f(x)$$

and $f(x) = 0$ since v is independent of x at $y = \pm b$ and $v = 0$ at $y = 0$.

From (4.1b)

$$c = \frac{3W}{4b^3}$$

therefore

$$u = \frac{3W}{4b^3} x(y^2 - b^2) \quad (4.3)$$

and

$$v = \frac{3W}{4b^3} y(b^2 - \frac{y^2}{3}) \quad (4.4)$$

Equations (4.3) and (4.4) satisfy the Navier Stokes equations since $\text{curl} (\nabla^2 \vec{U}) = 0$. Substitution of (4.3) and (4.4) in the Navier Stokes equations (2.10) gives

$$\frac{\partial p}{\partial x} = \frac{3}{2} \frac{W_{ux}}{b^3} \quad (4.5)$$

$$\frac{\partial p}{\partial y} = -\frac{3}{2} \frac{W_{uy}}{b^3} \quad (4.6)$$

Substitution of equations (4.5) and (4.6) in

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy$$

gives the exact differential

$$dp = \frac{3W_{ux}}{2b^3} (x dx - y dy)$$

and integration gives

$$p = \frac{3W_{ux}}{4b^3} (x^2 - y^2 + c_1)$$

At $x = \pm a$, $y = \pm b$ take $p = 0$

then

$$c_1 = -a^2 + b^2$$

and

$$p = \frac{3W\mu}{4b^3} (x^2 - y^2 - a^2 + b^2) \quad (4.7)$$

and at $x = \pm a$

$$p = -\frac{3W\mu}{4b^3} (y^2 - b^2) .$$

The normal stress component in the x-direction can be written as

$$\sigma_x = s_x - p$$

and from equation (2.5)

$$s_x = 2\mu \frac{\partial u}{\partial x} .$$

The partial derivative with respect to y of the normal stress is

$$\frac{\partial \sigma_x}{\partial y} = \frac{\partial s_x}{\partial y} - \frac{\partial p}{\partial y} \quad (4.8)$$

and

$$\frac{\partial s_x}{\partial y} = 2\mu \frac{\partial^2 u}{\partial x \partial y} . \quad (4.9)$$

Substituting (4.3) in (4.9) gives

$$\frac{\partial s_x}{\partial y} = \frac{3\mu W y}{b^3}$$

and substituting this equation and (4.6) in (4.8) gives

$$\frac{\partial \sigma_x}{\partial y} = \frac{9\mu W y}{2b^3} \quad \text{at } x = \pm a . \quad (4.10)$$

The shear stress is

$$\tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (4.11)$$

and using (4.3) and (4.4) this gives

$$\tau_{xy} = \frac{3\mu W_{xy}}{2b^3}$$

and at $x = a$

$$\tau_{xy} = \frac{3\mu}{2b^3} W_{ay} \quad (4.12)$$

The distribution of stress indicated by the elementary solution is somewhat artificial. A more realistic boundary condition at $x = \pm a$ would be constant normal and zero shear stress. Since the governing equations are linear an auxiliary solution can be superimposed on this solution and thereby satisfy the conditions of constant normal stress and zero shear stress at the edges.

4.3 Auxiliary Solution

The range $(-a \leq x \leq a, -b \leq y \leq b)$ can be converted to the range $(-\frac{\pi}{2} \leq X \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq Y \leq \frac{\pi}{2})$ with the substitutions

$$x = \frac{2aX}{\pi}, \quad y = \frac{2bY}{\pi} \quad (4.13)$$

Using the dimensionless stream function

$$\Psi = \frac{\psi}{aW} \quad (4.14)$$

the biharmonic equation (2.13) becomes

$$\frac{\partial^4 \Psi}{\partial X^4} + 2\alpha^2 \frac{\partial^4 \Psi}{\partial X^2 \partial Y^2} + \alpha^4 \frac{\partial^4 \Psi}{\partial Y^4} = 0 \quad (4.15)$$

where $\alpha = \frac{a}{b}$ is the ratio of the plate width to separation distance.

Assume that Ψ can be expanded in a double Fourier sine series as follows

$$\Psi = \sum_r \sum_s A_{rs} \sin rX \sin sY \quad \begin{array}{l} -\frac{\pi}{2} \leq X \leq \frac{\pi}{2} \\ -\frac{\pi}{2} \leq Y \leq \frac{\pi}{2} \end{array} \quad (4.16)$$

where r and s are odd positive integers because of the symmetry of the problem

$$\frac{\partial \Psi}{\partial Y} = \sum_r \sum_s s A_{rs} \sin rX \cos sY \quad -\frac{\pi}{2} \leq X \leq \frac{\pi}{2}$$

$$-\frac{\pi}{2} \leq Y \leq \frac{\pi}{2}$$

$$\frac{\partial^2 \Psi}{\partial Y^2} = - \sum_r \sum_s s^2 A_{rs} \sin rX \sin sY \quad -\frac{\pi}{2} \leq X \leq \frac{\pi}{2}$$

$$-\frac{\pi}{2} \leq Y \leq \frac{\pi}{2}$$

$$\frac{\partial^3 \Psi}{\partial Y^3} = - \sum_r \sum_s s^3 A_{rs} \sin rX \cos sY \quad -\frac{\pi}{2} \leq X \leq \frac{\pi}{2}$$

$$-\frac{\pi}{2} < Y < \frac{\pi}{2}$$

$$\frac{\partial^4 \Psi}{\partial Y^4} = \sum_r \sum_s \left(s^4 A_{rs} + (-1)^{\frac{s-1}{2}} b_r \right) \sin rX \sin sY \quad \begin{array}{l} -\frac{\pi}{2} \leq X \leq \frac{\pi}{2} \\ -\frac{\pi}{2} \leq Y \leq \frac{\pi}{2} \end{array} \quad (4.17a)$$

$$\frac{\partial \Psi}{\partial X} = \sum \sum r A_{rs} \cos rX \sin sY \quad -\frac{\pi}{2} < X < \frac{\pi}{2}$$

$$-\frac{\pi}{2} \leq Y \leq \frac{\pi}{2}$$

$$\frac{\partial^2 \Psi}{\partial X^2} = \sum \sum \left(-r^2 A_{rs} + (-1)^{\frac{r-1}{2}} c_s \right) \sin rX \quad -\frac{\pi}{2} \leq X \leq \frac{\pi}{2}$$

$$-\frac{\pi}{2} \leq Y \leq \frac{\pi}{2}$$

$$\frac{\partial^3 \Psi}{\partial X^3} = \sum \sum \left(-r^3 A_{rs} + r(-1)^{\frac{r-1}{2}} c_s \right) \cos rX \quad -\frac{\pi}{2} < X < \frac{\pi}{2}$$

$$-\frac{\pi}{2} \leq Y \leq \frac{\pi}{2}$$

$$\frac{\partial^4 \Psi}{\partial X^4} = \sum \sum \left(r^4 A_{rs} - r^2 (-1)^{\frac{r-1}{2}} c_s + (-1)^{\frac{r-1}{2}} d_s \right) \sin rX \sin sY \quad -\frac{\pi}{2} \leq X \leq \frac{\pi}{2}$$

$$-\frac{\pi}{2} \leq Y \leq \frac{\pi}{2} \quad (4.17b)$$

$$\frac{\partial^4 \Psi}{\partial X^2 \partial Y^2} = \sum \sum \left(r^2 s^2 A_{rs} - s^2 c_s (-1)^{\frac{r-1}{2}} \right) \sin rX \sin sY \quad -\frac{\pi}{2} \leq X \leq \frac{\pi}{2}$$

$$-\frac{\pi}{2} \leq Y \leq \frac{\pi}{2} \quad (4.17c)$$

$$\text{where } b_r = \frac{16}{\pi^2} \int_0^{\frac{\pi}{2}} \frac{\partial^3 \Psi}{\partial Y^3} \left(X, \frac{\pi}{2} \right) \sin sY \, dY \quad (4.17d)$$

$$c_s = \frac{16}{\pi^2} \int_0^{\frac{\pi}{2}} \frac{\partial \Psi}{\partial X} \left(\frac{\pi}{2}, Y \right) \sin sY \, dY \quad (4.17e)$$

$$d_s = \frac{16}{\pi^2} \int_0^{\frac{\pi}{2}} \frac{\partial^3 \Psi}{\partial X^3} \left(\frac{\pi}{2}, Y \right) \sin sY \, dY \quad (4.17f)$$

Substituting equations (4.17a,b,c) in (4.15) gives

$$\begin{aligned}
 (r^2 + \alpha^2 s^2) A_{rs} + \alpha^4 (-1)^{\frac{s-1}{2}} b_r - (-1)^{\frac{r-1}{2}} \left[(r^2 + 2\alpha^2 s^2) \right. \\
 \left. c_s - d_s \right] = 0
 \end{aligned}$$

$$A_{rs} = \frac{-\alpha^4 (-1)^{\frac{s-1}{2}} b_r + (-1)^{\frac{r-1}{2}} \left[(r^2 + 2\alpha^2 s^2) c_s - d_s \right]}{(r^2 + \alpha^2 s^2)^2} \quad (4.18)$$

This expression is valid for the plane flow of a liquid.

The boundary conditions for the auxiliary problem are:

$$\begin{aligned}
 U = \frac{u}{w} = 0 \quad \text{at } Y = \pm \frac{\pi}{2} \\
 V = \frac{v}{w} = 0 \quad \text{at } Y = \pm \frac{\pi}{2}
 \end{aligned} \quad (4.18a,b)$$

and making the stress conditions for the auxiliary solution the negative of the stress conditions for the elementary solution equations (4.10) and (4.12) yield

$$\begin{aligned}
 \frac{\partial \sigma_x}{\partial Y} &= \frac{-18\mu WY}{\pi b^2} \quad \text{at } X = \pm \frac{\pi}{2} \\
 \tau_{xy} &= \frac{-3\alpha\mu WY}{\pi b} \quad \text{at } X = \frac{\pi}{2}
 \end{aligned} \quad (4.18c,d)$$

The boundary condition (4.18a) is satisfied by the elementary solution and use of (4.18b) and

$$v = -\frac{\pi}{2} \sum \sum r A_{rs} \cos rX \sin sY \quad -\frac{\pi}{2} < X < \frac{\pi}{2}$$

gives

$$\sum_r r \left(\sum_s A_{rs} (-1)^{\frac{s-1}{2}} \right) \cos rX = 0$$

which is true for all r .

Therefore,

$$\sum_s A_{rs} (-1)^{\frac{s-1}{2}} = 0 \quad -\frac{\pi}{2} < X < \frac{\pi}{2} \quad (4.19)$$

Equation (4.17e) yields

$$V = -\frac{\pi^2}{8} \sum_s c_s \sin sY \quad \text{at } X = \pm \frac{\pi}{2}$$

and the use of (4.18b) gives

$$\sum_s c_s (-1)^{\frac{s-1}{2}} = 0 \quad \text{at } X = \pm \frac{\pi}{2} \quad (4.19a)$$

The boundary condition (4.18b) is satisfied by (4.19) and (4.19a).

Substituting the components of velocity derived from the stream function (2.12) into equation (4.8) gives

$$\frac{\partial \sigma_x}{\partial Y} = -\frac{\pi^2 \mu W}{4b} \left(\frac{1}{\alpha^2} \frac{\partial^3 \Psi}{\partial X^3} - \frac{\partial^3 \Psi}{\partial X \partial Y^2} \right) \quad (4.20)$$

Equations (4.17e,f) yield

$$\frac{\partial \Psi}{\partial X} = \sum_s \frac{\pi}{4} c_s \sin sY \quad \text{at } X = \pm \frac{\pi}{2}$$

$$\frac{\partial^3 \Psi}{\partial X^3} = \sum_s \frac{\pi}{4} d_s \sin sY \quad \text{at } X = \pm \frac{\pi}{2}$$

Hence (4.20) becomes at the edges

$$\frac{\partial \sigma_x}{\partial Y} = -\frac{\pi^3 \mu W}{16b} \sum_s \left(\frac{d_s}{\alpha^2} + s^2 c_s \right) \sin sY . \quad (4.21)$$

Expanding the boundary condition (4.18) in a Fourier sine series gives

$$\frac{-18\mu WY}{\pi^2 b} = \frac{-72\mu W}{\pi^3 b} \sum_s \frac{(-1)^{\frac{s-1}{2}}}{s^2} \sin sY . \quad (4.22)$$

Equating (4.21) and (4.22) yields

$$d_s + \alpha^2 s^2 c_s - \frac{1152}{\pi^6} \alpha \frac{(-1)^{\frac{s-1}{2}}}{s^2} = 0 .$$

Eliminating d_s from equation (4.18)

$$A_{rs} = \frac{-\alpha^4 (-1)^{\frac{s-1}{2}} b_r}{(r^2 + \alpha^2 s^2)^2} + \frac{(-1)^{\frac{r-1}{2}} \left((r^2 + 3\alpha^2 s^2) c_s - \alpha^2 \frac{1152}{\pi^6} \frac{(-1)^{\frac{s-1}{2}}}{s^2} \right)}{(r^2 + \alpha^2 s^2)^2} \quad (4.23)$$

Similarly the shear stress condition (4.11) becomes

$$\tau_{xy} = \frac{\pi^2 \mu W}{4b} \left(\frac{1}{\alpha} \frac{\partial^2 \Psi}{\partial X^2} - \alpha \frac{\partial^2 \Psi}{\partial Y^2} \right) .$$

Substitution gives

$$\tau_{xy} = \frac{\pi^2 \mu W}{4b} \sum \sum \left(\frac{r^2 A_{rs}}{\alpha^2} - \frac{(-1)^{\frac{r-1}{2}}}{\alpha} c_s - \alpha^2 s^2 A_{rs} \right) \frac{(-1)^{\frac{r-1}{2}}}{s^2} \sin sY . \quad (4.24)$$

Expanding the boundary condition (4.18d) gives

$$\frac{-3\alpha\mu}{\pi b} \frac{WY}{W} = \frac{-12\alpha\mu}{\pi^2 b} \sum \frac{(-1)^{\frac{s-1}{2}}}{s^2} \sin sY \quad (4.25)$$

Equating (4.24) and (4.25) yields

$$\sum_r (-1)^{\frac{r-1}{2}} (r^2 - \alpha^2 s^2) A_{rs} - c_s + \frac{48}{\pi^4} \alpha^2 \frac{(-1)^{\frac{s-1}{2}}}{s^2} = 0 \quad (4.26)$$

Substitution of (4.23) gives

$$\sum_r (-1)^{\frac{r+s-2}{2}} \frac{\alpha^4 (r^2 - \alpha^2 s^2) b_r}{(r^2 + \alpha^2 s^2)^2} + \frac{4\alpha^2 s^4 c_s}{(r^2 + \alpha^2 s^2)^2} = \frac{48\alpha^2}{\pi^4} \frac{(-1)^{\frac{s-1}{2}}}{s^2} \left(1 - \frac{24}{\pi^2} \sum_r \frac{(r^2 - \alpha^2 s^2)}{(r^2 + \alpha^2 s^2)^2} \right) \quad (4.27)$$

Substitution of (4.23) in (4.18) gives

$$\sum_s \frac{\alpha^4 b_r}{(r^2 + \alpha^2 s^2)^2} - \frac{(-1)^{\frac{r+s-2}{2}} (r^2 + 3\alpha^2 s^2) c_s}{(r^2 + \alpha^2 s^2)^2} = -(-1)^{\frac{r-1}{2}}$$

$$\frac{1156}{\pi^6} \alpha^2 \sum_s \frac{1}{s^2 (r^2 - \alpha^2 s^2)^2} \quad (4.28)$$

$$\text{and} \quad \sum_s c_s (-1)^{\frac{s-1}{2}} = 0 \quad (4.19a)$$

Equations (4.27), (4.28) and (4.19a) are sufficient to determine the sets of b_r and c_s from which the set of coefficients

A_{rs} can be determined.

The effect of the auxiliary solution on the total force required to maintain the constant relative normal motion of the two plates can be found as follows. Since the plates represent a plane of maximum shear the normal stresses are equal and the stress deviator is zero.

Therefore

$$\sigma_x = \sigma_y = -p$$

The Navier Stokes equations (2.10) become on substitution

$$\frac{\partial p}{\partial x} = \frac{\pi^2 \mu W}{4b} \left(\frac{\partial^3 \Psi}{\partial x^2 \partial y} + \alpha^2 \frac{\partial^3 \Psi}{\partial y^3} \right) \quad (4.29)$$

$$\frac{\partial p}{\partial y} = \frac{-\pi^2 \mu W}{4b} \left(\frac{1}{\alpha^2} \frac{\partial^3 \Psi}{\partial x^2} + \frac{\partial^3 \Psi}{\partial x \partial y^2} \right)$$

Substitution of the appropriate derivatives of (4.17) and setting $y = \frac{\pi}{2}$ gives

$$\frac{\partial p}{\partial y} = 0 \quad \text{at} \quad y = \frac{\pi}{2}.$$

Since $U = V = 0$ at $y = \frac{\pi}{2}$

$$\frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial x^2} = \dots = 0 \quad \text{at} \quad y = \frac{\pi}{2}$$

and equations (4.29) yield

$$\frac{dp}{dX} = \frac{\pi^2 \alpha^2 \mu W}{4b} \frac{\partial^3 \Psi}{\partial Y^3} \quad \text{at } Y = \frac{\pi}{2}$$

Substituting in (4.29) and integrating yields

$$p = \frac{-\pi^3 \alpha^3 \mu W}{16b} \sum_r \frac{b_r}{r} \cos rX \quad \text{at } Y = \frac{\pi}{2}$$

which satisfies the condition of zero pressure at $X = \frac{\pi}{2}$.

CHAPTER V

CONCLUSION

5.1 Summary of Results

The equations (4.19), (4.27) and (4.28) were set up in a program to be run on the IBM 7040 computer. This involved finding the coefficients of the unknowns b_r and c_s and arranging them in a suitable form for solution of the simultaneous equations. The program was written in double precision to obtain sufficient accuracy when a large number of equations were used. The subroutine Jordan, which is available in the Department of Computing Science was used to solve the equations.

The problem was first solved satisfying only the normal stress and normal velocity conditions at the edges (equations (4.19) and (4.28)). A possible condition which satisfied the normal stress condition at the edges was to assume all the c_s coefficients to be zero and solve equation (4.28) for the coefficients b_s . The shear stress condition (4.27) was then not satisfied and the shear stress for the combined problem remained approximately equal to the condition for the elementary solution.

The effect of adding the auxiliary solution is to increase the X velocity component in the neighborhood of the midpoint of the two plates and decrease the velocity near the plates. This effect dies out rapidly for increased

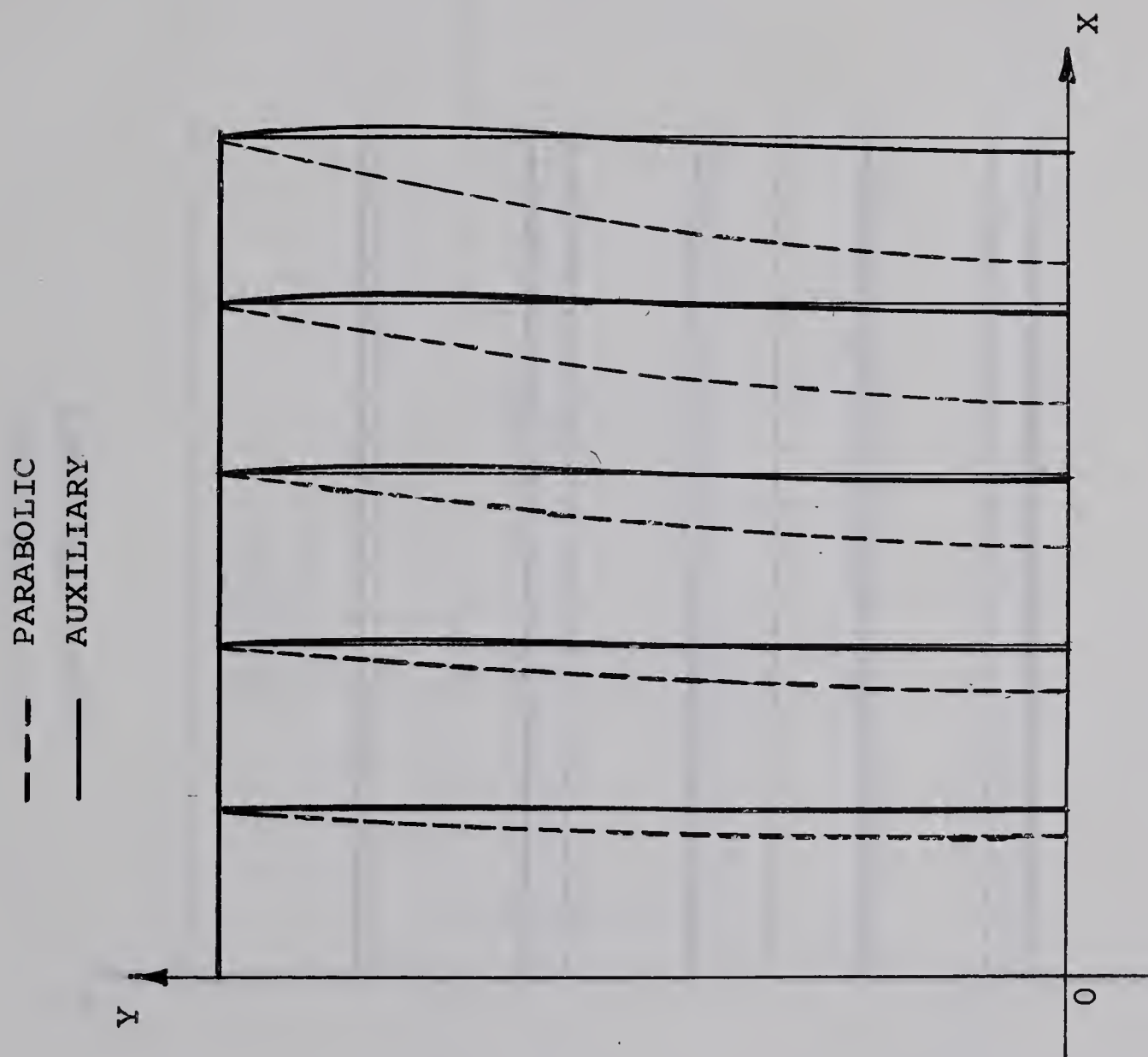


FIGURE 5.1 X VELOCITY PROFILES FOR PLATE WIDTH/SEPARATION = 1

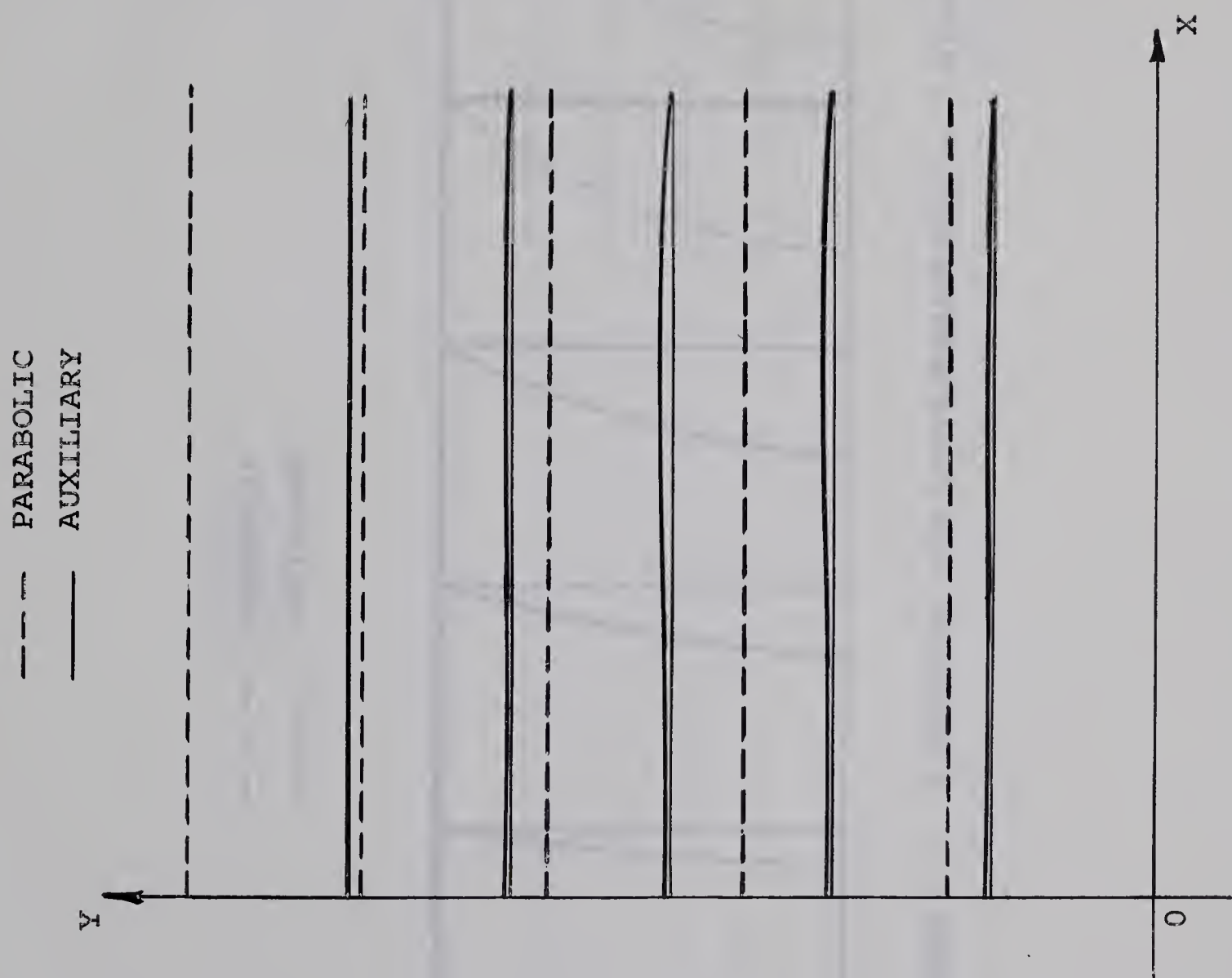


FIGURE 5.2 Y VELOCITY PROFILES FOR WIDTH/SEPARATION = 1

- - - PARABOLIC
 — AUXILIARY

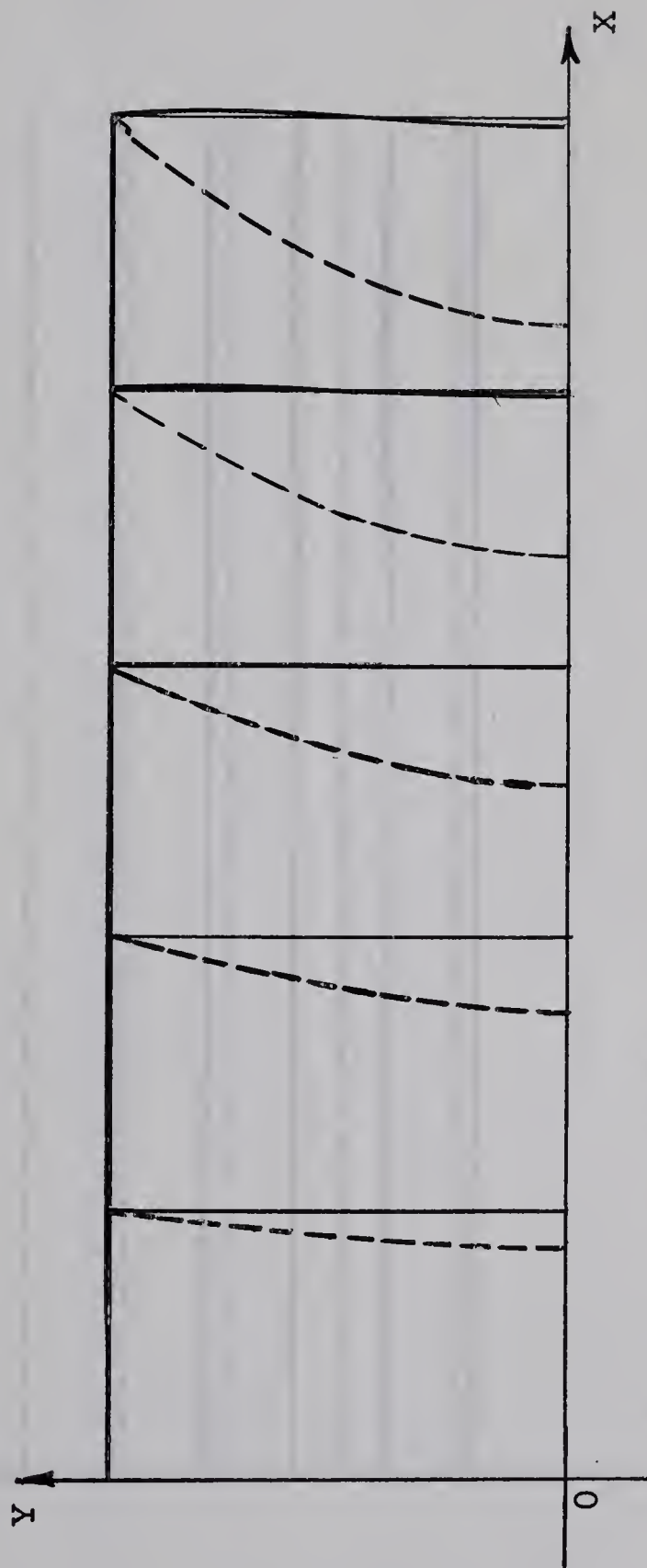


FIGURE 5.3 X VELOCITY FOR WIDTH/SEPARATION = 3 WITH COEFFICIENTS $c_s = 0$

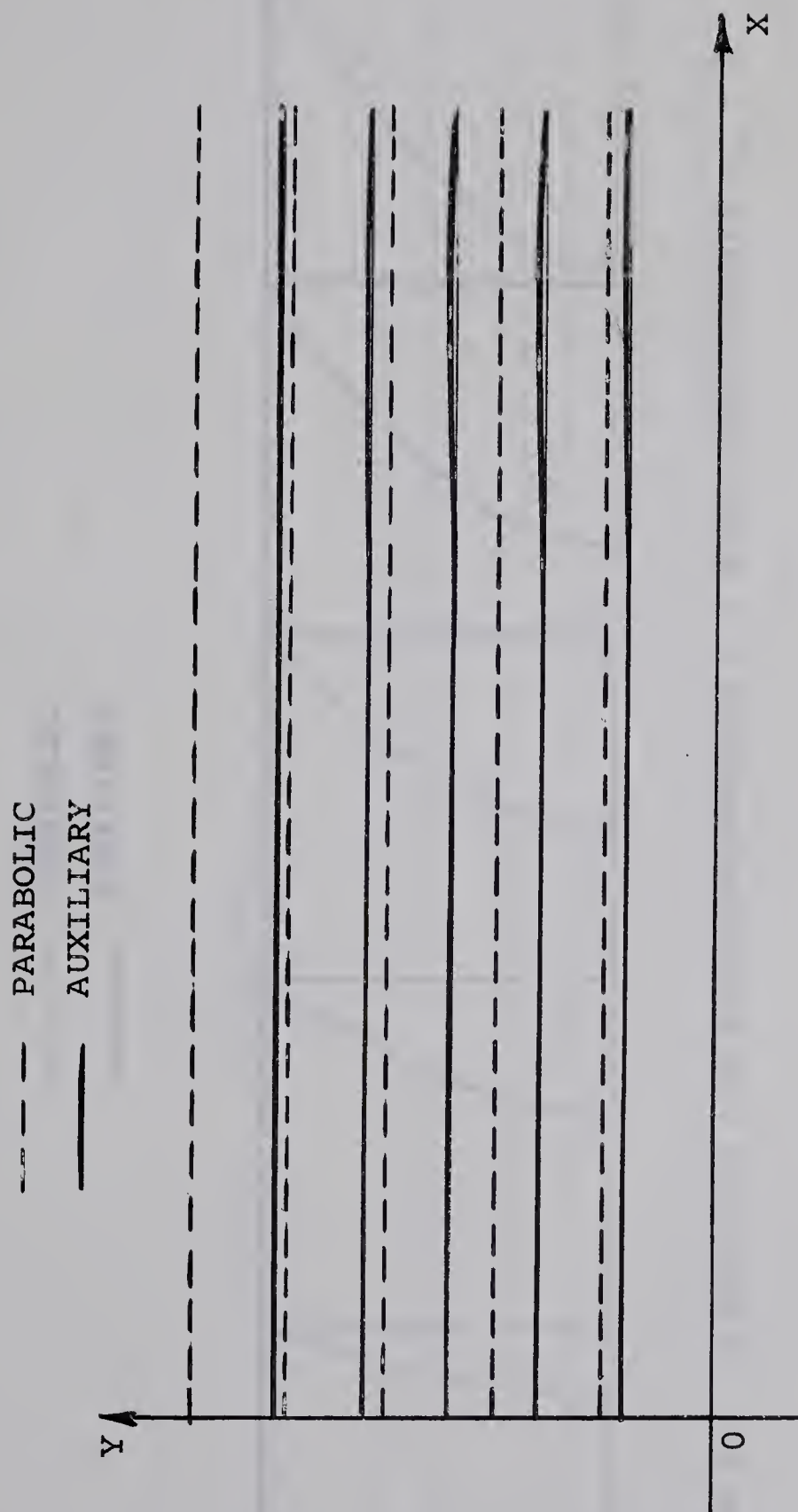


FIGURE 5.4 Y VELOCITY FOR WIDTH/SEPARATION = 3 WITH COEFFICIENTS $c_s = 0$

- - - PARABOLIC
 ——— AUXILIARY

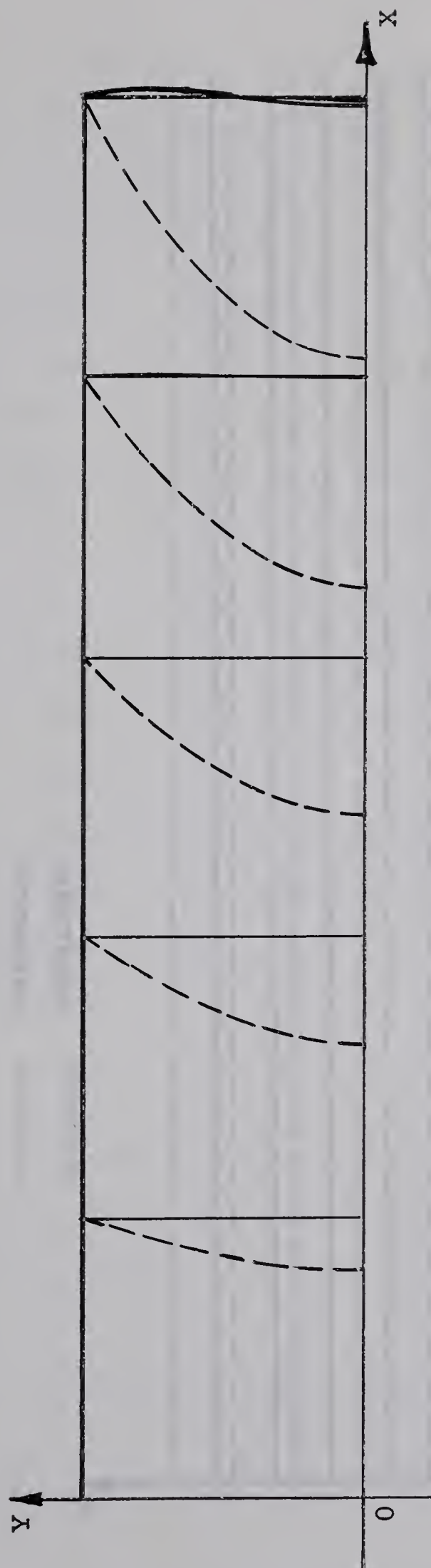


FIGURE 5.5 X VELOCITY FOR WIDTH/SEPARATION = 5 WITH COEFFICIENTS $c_s = 0$

--- PARABOLIC
 — AUXILIARY

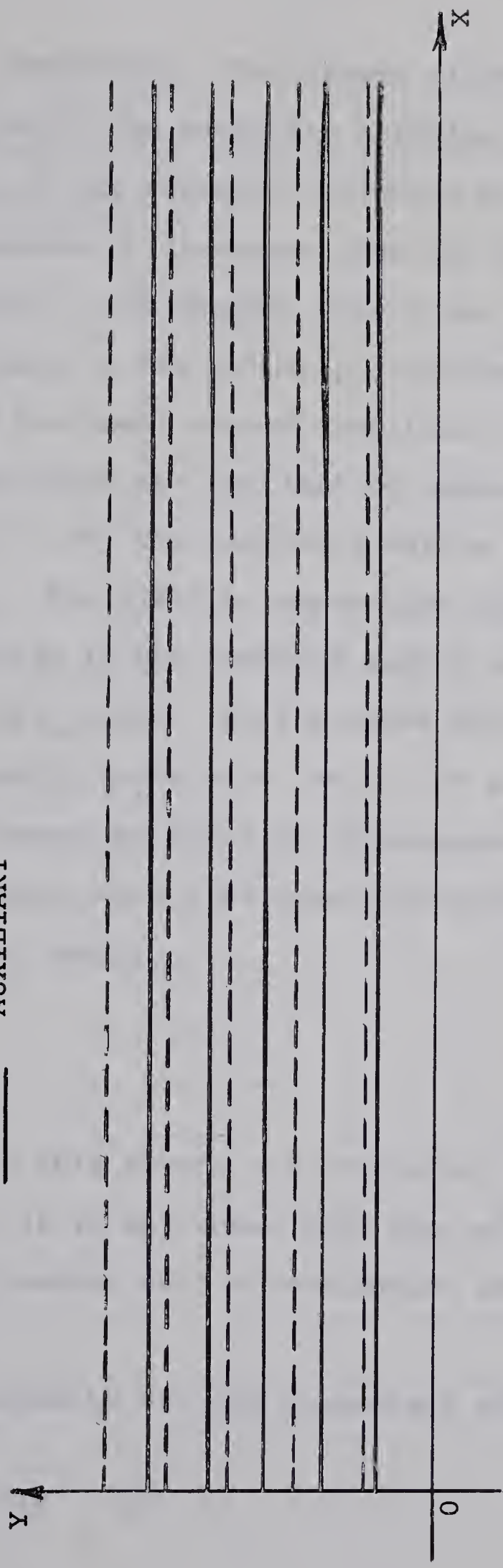


FIGURE 5.6 Y VELOCITY FOR WIDTH SEPARATION = 5 WITH COEFFICIENTS $c_s = 0$

ratio of plate width to separation. For larger plate width to separation the addition of the auxiliary solution produced only small changes in the classical solution and these changes were negligible at distances from the edge greater than the separation. The maximum effect was approximately a ten percent change in the velocity distribution.

In order to satisfy the shear stress condition (4.27) equation (4.19) was substituted for the last of equations (4.27) and with equation (4.28) the resulting matrix was solved for the c_s and b_s . For width to separation equal to one the velocity profiles in the interior region agreed with the solution with the c_s zero. At the edges the velocities did not form a smooth curve with the c_s not equal to zero and is possibly caused by the slow convergence of the solution. The results were also unsatisfactory for larger width to separation ratios.

5.2 Discussion

The problem solved in this thesis has completely ignored inertia effects. It is now shown that the velocity components found by making this approximation satisfy this approximation.

The x component of velocity for the elementary solution is

$$u = \frac{3W}{4b^3} x(y^2 - b^2) .$$

Substituting this component in the material derivative and the viscous term and letting

$$\bar{x} = \frac{x}{b} \quad \text{and} \quad \bar{y} = \frac{y}{b}$$

it is found that

$$\frac{\rho \frac{du}{dt}}{\mu \nabla^2 u} = \frac{\rho W b}{\mu} \left(-\frac{3}{4} (\bar{y}^2 - 1) - \frac{1}{2} + \frac{3}{8} (\bar{y}^2 - 1)^2 + \frac{3}{4} \bar{x} \bar{y} (\bar{y}^2 - 1) \right).$$

A similar result can be found for the y velocity and it can be seen, since the quantities in brackets are $O(1)$, that the neglect of the material derivative is justified for Reynolds number approaching zero.

In the one dimensional problem for radial flow between stationary plates J.L. Livesay⁽⁷⁾ has shown that the inertia effects can be significant at relatively low velocity. Unlike the problem considered the characteristic length was taken as the separation between the plates which was small compared with the radius of the plates. A method such as that suggested by Karman in this momentum-equation approach to boundary layer theory can be used to incorporate inertia effects. A velocity profile such as the one developed in Chapter IV can be assumed and the terms of the equation of motion integrated across the separation between the plates.

APPENDIX
COMPUTER PROGRAMS AND OUTPUT DATA
FOR VELOCITY FIELDS

The calculations for the velocity fields corresponding to various plate width to separation ratios was greatly facilitated by the use of an IBM 7040 digital computer. The solution of the large number of equations required was well suited as a computer problem. The Fortran source programs were designed for a general number of equations so that the effect of varying the number of equations could be examined.

The coefficients of the unknowns b_r and c_s in equations (4.27), (4.28) and (4.19) were arranged in matrix form suitable for the solution of simultaneous equations. The number of equations solved was taken equal to $2L$ allowing the determination of L of the unknowns b_r and L of the unknowns c_s . These equations were composed of L equations satisfying the normal velocity conditions (4.28), $L-1$ equations satisfying the shear stress equations and one equation satisfying the normal velocity condition at the edges. The subroutine Jordan available in the Department of Computing Science library was used to solve for the unknowns. The coefficients A_{rs} were then determined from equation (4.23) and the accuracy of the derivation and the

precision of the solution were then checked by substituting the A_{rs} into equations (4.18) and (4.23). The program and subroutine were written in double precision and only when a large number of equations were used was it found that the number of significant figures declined below a number acceptable for the problem.

FORTRAN SOURCE STATEMENT FOR THE VELOCITY FIELD

```

DOUBLE PRECISION A(50,51), PI,P,R,S,SUMR,SUMRR,SUMS,SUMSS,PT,PTP,
1 PR(10),PRP(10),X,Y,W,Z,UR(10),US(10),U(10),VR(10),VS(10),V(10),
1 UP(10),VP(10),C(25),RI(25),RTI(25),AI(25),ARS(50,50),CI(25),R(25)
INTEGER I,J,K,L,M,N,RHO,F,F,H
PI=3.141592653589793
DO 100 RHO=1,5,2
8 P=RHO
L=10
M=2*L+1
DO2 I=1,L
K=L+I
DO3 J=1,L
F=I+J-2
R=2*I-1
S=2*I-1
A(I,J)=(-1D0)**E*(R**2-(P*S)**2)/((R**2+(P*S)**2)**2)*P**4
A(L,J)=0D0
R=2*I-1
S=2*I-1
N=L+J
A(K,N)=-(-1D0)**F*(R**2+3D0*(P*S)**2)/((R**2+(P*S)**2)**2)
A(I,N)=0D0
F=J-1
A(L,N)=(-1D0)**F
3 A(K,J)=0D0
SUMR=0D0
SUMRR=0D0
SUMS=0D0
SUMSS=0D0
DO 15 J=1,L
S=2*I-1
R=2*I-1
SUMRR=SUMRR+(R**2-(P*S)**2)/((R**2+(P*S)**2)**2)
SUMR=SUMR+((R**2-(P*S)**2)*(R**2+3D0*(P*S)**2))/((R**2+(P*S)**2)**
12)
R=2*I-1
S=2*I-1
SUMSS=SUMSS+1D0/(S**2*(R**2+(P*S)**2)**2)
15 SUMS=SUMS+1D0/((R**2+(P*S)**2)**2)
F=I-1
R=2*I-1
S=2*I-1
A(I,K)=1D0-SUMR
A(K,I)=SUMS*P**4
A(I,M)=48D0/(PI**4)*P**2*(-1D0)**F/(S**2)*(1D0-24D0*SUMRR/(PI**2))
A(L,M)=0D0
N=I-1
A(K,M)=-1152D0/(PI**6)*P**2*SUMSS*(-1D0)**E
2 CONTINUE
N=2*L
F=L-1
A(L,N)=(-1D0)**E
M=2*L+1

```

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FBI FORM NO. 7-60 (REVISED 7-60)

GPO : 1960 O - 800-000-0000

```

      IT=6
      CALL JORDAN(N,M,A,IND,IT)
      IF(IND.EQ.0)GO TO 66
      WRITE(6,10)
10  FORMAT(24H THE MATRIX IS SINGULAR.)
      STOP
66  DO 60 I=1,L
      K=L+I
      B(I)=A(I,M)
60  C(K)=A(K,M)
      6 WRITE(6,11) P
11  FORMAT(1H1,3H P= D8.2,27H          B(R)          )
      WRITE(6,12)(B(K),K=1,L)
12  FORMAT(1H, 18X,D24.16)
      E=L+1
      WRITE(6,33)
33  FORMAT(1HK,39H          C(S)          )
      WRITE(6,12)(C(K),K=E,N)
      DO16 I=1,L
      R=2*I-1
      F=I-1
      DO17 J=1,L
      S=2*J-1
      K=J+L
      F=J-1
17  ARS(I,J)=(-P**4*(-1D0)**F*B(I)+(-1D0)**F*((R**2+3D0*(P*S)**2)*C(K)
1  -P**2*1152D0/(PI**6)*(-1D0)**F/(S**2)))/((R**2+(P*S)**2)**2)
16  CONTINUE
36  FORMAT(1HK,58H          SUMREQ
1SUMSEQ)
      WRITE(6,36)
      DO 34 I=1,L
      SUMR=0D0
      SUMS=0D0
      F=I-1
      K=I+L
      S=2*I-1
      DO 35 J=1,L
      F=J-1
      R=2*J-1
      SUMR =SUMR+(-1D0)**F*(R**2-(P*S)**2)*ARS(J,I)
35  SUMS=SUMS+(-1D0)**F*ARS(I,J)
      SUMR=-SUMR+C(K) -48D0/(PI**4)*P**2*(-1D0)**F/(S**2)
37  FORMAT(1H, 18X,2D24.16)
34  WRITE(6,37) SUMR,SUMS
      SUMS=0D0
      DO 39 I=1,L
      F=I-1
      K=L+I
39  SUMS=SUMS+C(K)*(-1D0)**F
      WRITE(6,40)
40  FORMAT(1HK,23H          SUMC(S)**(S-1)/2)
      WRITE(6,37) SUMS
      WRITE(6,13) P
13  FORMAT(1H3,3H P= F6.2,38H          THE COEFFICIENTS ARS C(S).NE.0)
      DO 14 I=1,5
26  FORMAT(1HK,5F12.6)

```



```

14 WRITE(6,26)(ARS(I,J),J=1,5)
    CALCULATION OF THE X - VELOCITY COMPONENT
    WRITE(6,22) P
22 FORMAT(1H2,3H P= F6.2,23H X - VELOCITY )
    WRITE(6,28)
28 FORMAT(1HK, 70H Y=0 PI/20 2PI/20 3PI/20 4PI/20 5PI/20 6PI/20 7
1PI/20 8PI/20 9PI/20)
    X=-PI/10D0
    DO18 N=1,6
    E=N-1
    WRITE(6,29) E
29 FORMAT(1HK,2HX=I1,5HPI/10)
    X=X+PI/10D0
    Y=-PI/20D0
    DO19 K=1,10
    Y=Y+PI/20D0
    US(K)=0D0
    DO20 I=1,L
    UR(K)=0D0
    DO21 J=1,L
    R=2*J-1
    IF(N.LT.6)GO TO 24
    E=J-1
    UR(K)=UR(K)+ARS(J,I)*(-1D0)**E
    GO TO 21
24 UR(K)=UR(K)+ARS(J,I)*SIN(R*X)
21 CONTINUE
    S=2*I-1
20 US(K)=US(K)+UR(K)*S*COS(S*Y)
    UP(K)=6D0*P*X*((Y/PI)**2-.25D0)/PI
19 U(K)= PI/2D0*P*US(K)
    WRITE(6,25)(U(K),K=1,10)
23 FORMAT(1HJ,10D13.5)
25 FORMAT(1HJ,10F7.3)
    WRITE(6,30)
30 FORMAT(1HJ,40H PARABOLIC )
18 WRITE(6,25)(UP(K),K=1,10)
    CALCULATION OF THE Y - VELOCITY COMPONENT
    WRITE(6,31)
31 FORMAT(1H2,35H Y - VELOCITY )
    WRITE(6,82)
82 FORMAT(1HK, 70H X=0 PI/20 2PI/20 3PI/20 4PI/20 5PI/20 6PI/20 7
1PI/20 8PI/20 9PI/20)
    Y=-PI/10D0
    DO 78 N=1,6
    E=N-1
    WRITE(6,79) E
79 FORMAT(1HK,2HY=I1,5HPI/10)
    Y=Y+PI/10D0
    X=-PI/20D0
    DO 80 K=1,10
    X=X+PI/20D0
    VR(K)=0D0
    DO 70 I=1,L
    VS(K)=0D0
    DO 71 J=1,L
    S=2*J-1

```



```

71 VS(K)=VS(K)+ARS(I,J)*SIN(S*Y)
   R=2*I-1
70 VR(K)=VR(K)+VS(K)*R*COS(R*X)
   VP(K)=6D0*Y*(.25D0-(Y/PI)**2/3D0)/PI
80 V(K)=-PI/2D0*VR(K)
   WRITE(6,25)(V(K),K=1,10)
   WRITE(6,30)
78 WRITE(6,25)(VP(K),K=1,10)
   Y=-PI/10D0
DO 85 N=1,6
   Y=Y+PI/10D0
   V(N)=0D0
DO 50 I=1,L
   K=L+I
   S=2*I-1
50 V(N)=V(N)+C(K)*SIN(S*Y)
85 CONTINUE
   WRITE(6,51)
51 FORMAT(1HK,24H      Y-VELOCITY AT X=PI/2)
   WRITE(6,23)(V(N),N=1,6)
100 CONTINUE
7 STOP
END

```

[illegible]

SUBROUTINE JORDAN(N,M,A,IND,IT)

DOUBLE PRECISION A(50,51),P,B

INTEGER LR(50),LC(51)

C M IS NO. OF COLUMNS

C N IS NO. OF ROWS

C A IS MATRIX

C IF IND IS ZERO, INVERSION IS GIVEN

C IF IND IS ONE, MATRIX IS SINGULAR

C IT IS OUTPUT UNIT

3 KZ=1

DO 16 K=1,N

LR(K)=K

LC(K)=K

I=K

IZ=K

P=DABS(A(K,K))

DO 5 J=K,N

DO 5 JZ=K,N

IF(P.GE.DABS(A(J,JZ)))GO TO 5

P=DABS(A(J,JZ))

I=J

IZ=JZ

5 CONTINUE

IF(I.GT.K)GO TO 23

6 IF(IZ.GT.K)GO TO 18

7 IF(P.GE.1.0000000000000000D-24)GO TO 12

WRITE(IT,42)

IND=1

RETURN

12 P=1.0000000000000000D0/A(K,K)

A(K,K)=1.0000000000000000D0

DO 13 J=1,M

13 A(K,J)=P*A(K,J)

DO 16 I=1,N

IF(K.EQ.I)GO TO 16

14 B=A(I,K)

A(I,K)=0.0000000000000000D0

DO 15 J=1,M

15 A(I,J)=A(I,J)-B*A(K,J)

16 CONTINUE

KZ=0

K=N

DO 26 IA=1,N

IF(LR(K).EQ.K)GO TO 21

17 IZ=LR(K)

C CHANGE COLUMNS IZ AND K

18 DO 19 J=1,N

B=A(J,IZ)

A(J,IZ)=A(J,K)

19 A(J,K)=B

IF(KZ.EQ.0)GO TO 21

20 LC(K)=IZ

GO TO 12

21 IF(LC(K).EQ.K)GO TO 26

22 I=LC(K)


```
C      CHANGE ROWS I AND K
23      DO 24 J=1,M
        B=A(I,J)
        A(I,J)=A(K,J)
24      A(K,J)=B
        IF(KZ.EQ.0)GO TO 26
25      LR(K)=I
        GO TO 6
26      K=K-1
        IND=0
        RETURN
42      FORMAT(24H THE MATRIX IS SINGULAR.)
        END
```


P= 1.00 THE COEFFICIENTS ARE C(S).NE.0

-0.016156 -0.011838 0.003941 0.000303 0.000004

0.006189 0.003445 -0.002384 -0.000273 0.000008

-0.002461 -0.000904 0.001201 0.000215 -0.000016

0.001271 0.000294 -0.000626 -0.000158 0.000018

-0.000779 -0.000108 0.000351 0.000119 -0.000020

P= 1.00 X - VELOCITY

Y=0 | $\pi/20$ $2\pi/20$ $3\pi/20$ $4\pi/20$ $5\pi/20$ $6\pi/20$ $7\pi/20$ $8\pi/20$ $9\pi/20$

X=0 $\pi/10$

0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000

PARABOLIC

-0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000

X=1 $\pi/10$

-0.009 -0.009 -0.007 -0.004 -0.001 0.003 0.006 0.007 0.006 0.004

PARABOLIC

-0.150 -0.148 -0.144 -0.136 -0.126 -0.112 -0.096 -0.076 -0.054 -0.028

X=2 $\pi/10$

-0.022 -0.021 -0.017 -0.011 -0.003 0.006 0.013 0.017 0.016 0.010

PARABOLIC

-0.300 -0.297 -0.288 -0.273 -0.252 -0.225 -0.192 -0.153 -0.108 -0.057

X=3 $\pi/10$

-0.039 -0.039 -0.034 -0.027 -0.011 0.007 0.026 0.036 0.036 0.022

PARABOLIC

-0.450 -0.445 -0.432 -0.409 -0.378 -0.337 -0.288 -0.229 -0.162 -0.085

X=4 $\pi/10$

-0.052 -0.056 -0.063 -0.059 -0.037 0.002 0.044 0.071 0.073 0.046

PARABOLIC

-0.600 -0.594 -0.576 -0.546 -0.504 -0.450 -0.384 -0.306 -0.216 -0.114

X=5 $\pi/10$

-0.167 0.070 -0.219 -0.001 -0.194 0.074 -0.012 0.171 0.068 0.083

PARABOLIC

-0.750 -0.742 -0.720 -0.682 -0.630 -0.562 -0.480 -0.382 -0.270 -0.142

Y - VELOCITY

X=0 PI/20 2PI/20 3PI/20 4PI/20 5PI/20 6PI/20 7PI/20 8PI/20 9PI/20

$$Y = OP I / 10$$

-0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000

PARABOLIC

0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000

$$Y = 1PI/10$$

0.009 0.007 0.011 0.010 0.016 0.016 0.022 0.017 0.019 -0.011

PARABOLIC

0.148 0.148 0.148 0.148 0.148 0.148 0.148 0.148 0.148 0.148

$$Y = 2PI/10$$

0.011 0.012 0.014 0.018 0.024 0.032 0.041 0.050 0.058 0.051

PARABOLIC

0.284 0.284 0.284 0.284 0.284 0.284 0.284 0.284 0.284 0.284

$$Y = 3PI/10$$

0.009 0.010 0.011 0.016 0.021 0.030 0.040 0.055 0.069 0.070

PARABOLIC

0.396 0.396 0.396 0.396 0.396 0.396 0.396 0.396 0.396 0.396

$$Y = 4PI/10$$

0.004 0.002 0.005 0.004 0.009 0.010 0.018 0.019 0.033 0.018

PARABOLIC

0.472 0.472 0.472 0.472 0.472 0.472 0.472 0.472 0.472 0.472

$$Y = 5PI/10$$
[illegible]

PARABOLIC

0.500 0.500 0.500 0.500 0.500 0.500 0.500 0.500 0.500 0.500

FORTRAN SOURCE STATEMENT FOR THE VELOCITY FIELD - CS=0

```

DOUBLE PRECISION A(50,51), PI,P,R,S,SUMR,SUMRR,SUMS,SUMSS,PT,PTP,
1 PR(10),PRP(10),X,Y,W,Z,UR(10),US(10),U(10),VR(10),VS(10),V(10),
1 UP(10),VP(10),C(25),BI(25),BII(25),AI(25),ARS(50,50),CI(25),P(25)
INTEGER I,J,K,L,M,N,RHO,E,F,H
PI=3.141592653589793
DO 100 RHO=1,5,2
P=RHO
L=10
DO2 I=1,L
K=L+I
DO3 J=1,L
E=I+J-2
R=2*I-1
S=2*I-1
A(I,J)=(-1D0)**E*(R**2-(P*S)**2)/((R**2+(P*S)**2)**2)*P**4
R=2*I-1
S=2*I-1
N=L+J
A(K,N)=-(-1D0)**E*(R**2+3D0*(P*S)**2)/((R**2+(P*S)**2)**2)
A(I,N)=0D0
3 A(K,J)=0D0
SUMR=0D0
SUMRR=0D0
SUMS=0D0
SUMSS=0D0
DO 15 J=1,L
S=2*I-1
R=2*I-1
SUMRR=SUMRR+(R**2-(P*S)**2)/((R**2+(P*S)**2)**2)
SUMR=SUMR+((R**2-(P*S)**2)*(R**2+3D0*(P*S)**2))/((R**2+(P*S)**2)**
12)
R=2*I-1
S=2*I-1
SUMSS=SUMSS+1D0/(S**2*(R**2+(P*S)**2)**2)
15 SUMS=SUMS+1D0/((R**2+(P*S)**2)**2)
E=I-1
R=2*I-1
S=2*I-1
A(I,K)=1D0-SUMR
A(K,I)=SUMS*P**4
M=2*L+1
A(I,M)=48D0/(PI**4)*P**2*(-1D0)**E/(S**2)*(1D0-24D0*SUMRR/(PI**2))
N=I-1
A(K,M)=-1152D0/(PI**6)*P**2*SUMSS*(-1D0)**E
2 CONTINUE
N=2*L
DO 50 I=1,L
K=L+I
B(I)=A(K,M)/A(K,I)
50 C(K)=0D0
WRITE(6,53)
53 FORMAT(1HK,72H SUMR 48D0/(PI
1**4*(-1D0)**E/(S**2))
DO 51 I=1,L

```

$$\begin{aligned} \psi(x, y, z) &= c_1 \sin z + c_2 \cos z \\ \varphi(x, y, z) &= [D_1 \cos z + D_2 \sin z] (c_3 \cos y + c_4 \sin y) \\ \psi(x, y, z) &= c_1 \sin z + c_2 \cos z \end{aligned}$$

(1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13. 14. 15. 16. 17. 18. 19. 20. 21. 22. 23. 24. 25. 26. 27. 28. 29. 30. 31. 32. 33. 34. 35. 36. 37. 38. 39. 40. 41. 42. 43. 44. 45. 46. 47. 48. 49. 50. 51. 52. 53. 54. 55. 56. 57. 58. 59. 60. 61. 62. 63. 64. 65. 66. 67. 68. 69. 70. 71. 72. 73. 74. 75. 76. 77. 78. 79. 80. 81. 82. 83. 84. 85. 86. 87. 88. 89. 90. 91. 92. 93. 94. 95. 96. 97. 98. 99. 100. 101. 102. 103. 104. 105. 106. 107. 108. 109. 110. 111. 112. 113. 114. 115. 116. 117. 118. 119. 120. 121. 122. 123. 124. 125. 126. 127. 128. 129. 130. 131. 132. 133. 134. 135. 136. 137. 138. 139. 140. 141. 142. 143. 144. 145. 146. 147. 148. 149. 150. 151. 152. 153. 154. 155. 156. 157. 158. 159. 160. 161. 162. 163. 164. 165. 166. 167. 168. 169. 170. 171. 172. 173. 174. 175. 176. 177. 178. 179. 180. 181. 182. 183. 184. 185. 186. 187. 188. 189. 190. 191. 192. 193. 194. 195. 196. 197. 198. 199. 200. 201. 202. 203. 204. 205. 206. 207. 208. 209. 210. 211. 212. 213. 214. 215. 216. 217. 218. 219. 220. 221. 222. 223. 224. 225. 226. 227. 228. 229. 230. 231. 232. 233. 234. 235. 236. 237. 238. 239. 240. 241. 242. 243. 244. 245. 246. 247. 248. 249. 250. 251. 252. 253. 254. 255. 256. 257. 258. 259. 260. 261. 262. 263. 264. 265. 266. 267. 268. 269. 270. 271. 272. 273. 274. 275. 276. 277. 278. 279. 280. 281. 282. 283. 284. 285. 286. 287. 288. 289. 290. 291. 292. 293. 294. 295. 296. 297. 298. 299. 300. 301. 302. 303. 304. 305. 306. 307. 308. 309. 310. 311. 312. 313. 314. 315. 316. 317. 318. 319. 320. 321. 322. 323. 324. 325. 326. 327. 328. 329. 330. 331. 332. 333. 334. 335. 336. 337. 338. 339. 340. 341. 342. 343. 344. 345. 346. 347. 348. 349. 350. 351. 352. 353. 354. 355. 356. 357. 358. 359. 360. 361. 362. 363. 364. 365. 366. 367. 368. 369. 370. 371. 372. 373. 374. 375. 376. 377. 378. 379. 380. 381. 382. 383. 384. 385. 386. 387. 388. 389. 390. 391. 392. 393. 394. 395. 396. 397. 398. 399. 400. 401. 402. 403. 404. 405. 406. 407. 408. 409. 410. 411. 412. 413. 414. 415. 416. 417. 418. 419. 420. 421. 422. 423. 424. 425. 426. 427. 428. 429. 430. 431. 432. 433. 434. 435. 436. 437. 438. 439. 440. 441. 442. 443. 444. 445. 446. 447. 448. 449. 450. 451. 452. 453. 454. 455. 456. 457. 458. 459. 460. 461. 462. 463. 464. 465. 466. 467. 468. 469. 470. 471. 472. 473. 474. 475. 476. 477. 478. 479. 480. 481. 482. 483. 484. 485. 486. 487. 488. 489. 490. 491. 492. 493. 494. 495. 496. 497. 498. 499. 500. 501. 502. 503. 504. 505. 506. 507. 508. 509. 510. 511. 512. 513. 514. 515. 516. 517. 518. 519. 520. 521. 522. 523. 524. 525. 526. 527. 528. 529. 530. 531. 532. 533. 534. 535. 536. 537. 538. 539. 540. 541. 542. 543. 544. 545. 546. 547. 548. 549. 550. 551. 552. 553. 554. 555. 556. 557. 558. 559. 560. 561. 562. 563. 564. 565. 566. 567. 568. 569. 570. 571. 572. 573. 574. 575. 576. 577. 578. 579. 580. 581. 582. 583. 584. 585. 586. 587. 588. 589. 590. 591. 592. 593. 594. 595. 596. 597. 598. 599. 600. 601. 602. 603. 604. 605. 606. 607. 608. 609. 610. 611. 612. 613. 614. 615. 616. 617. 618. 619. 620. 621. 622. 623. 624. 625. 626. 627. 628. 629. 630. 631. 632. 633. 634. 635. 636. 637. 638. 639. 640. 641. 642. 643. 644. 645. 646. 647. 648. 649. 650. 651. 652. 653. 654. 655. 656. 657. 658. 659. 660. 661. 662. 663. 664. 665. 666. 667. 668. 669. 670. 671. 672. 673. 674. 675. 676. 677. 678. 679. 680. 681. 682. 683. 684. 685. 686. 687. 688. 689. 690. 691. 692. 693. 694. 695. 696. 697. 698. 699. 700. 701. 702. 703. 704. 705. 706. 707. 708. 709. 710. 711. 712. 713. 714. 715. 716. 717. 718. 719. 720. 721. 722. 723. 724. 725. 726. 727. 728. 729. 730. 731. 732. 733. 734. 735. 736. 737. 738. 739. 740. 741. 742. 743. 744. 745. 746. 747. 748. 749. 750. 751. 752. 753. 754. 755. 756. 757. 758. 759. 760. 761. 762. 763. 764. 765. 766. 767. 768. 769. 770. 771. 772. 773. 774. 775. 776. 777. 778. 779. 780. 781. 782. 783. 784. 785. 786. 787. 788. 789. 790. 791. 792. 793. 794. 795. 796. 797. 798. 799. 800. 801. 802. 803. 804. 805. 806. 807. 808. 809. 810. 811. 812. 813. 814. 815. 816. 817. 818. 819. 820. 821. 822. 823. 824. 825. 826. 827. 828. 829. 830. 831. 832. 833. 834. 835. 836. 837. 838. 839. 840.

```

      K=L+I
      E=I-1
      S=2*I-1
      SUMR=0D0
      DO 52 J=1,L
      R=2*J-1
52  SUMR=SUMR+A(I,J)*B(J)+(-1D0)**E*1152D0/(PI**6)*(R**2-(P*S)**2)/((R
      1**2+(P*S)**2)**2)/(S**2)
      SUMS=48D0/(PI**4)*P**2*(-1D0)**E/(S**2)
51  WRITE(6,37) SUMR,SUMS
      6 WRITE(6,11) P
11  FORMAT(1H1,3H P= D8.2,27H B(R) )
      WRITE(6,12)(B(K),K=1,L)
12  FORMAT(1H, 18X,D24.16)
      E=L+1
      WRITE(6,33)
33  FORMAT(1HK,39H C(S) )
      WRITE(6,12)(C(K),K=E,N)
      DO16 I=1,L
      R=2*I-1
      E=I-1
      DO17 J=1,L
      S=2*J-1
      K=J+L
      F=J-1
17  ARS(I,J)=(-P**4*(-1D0)**F*B(I)+(-1D0)**F*((R**2+3D0*(P*S)**2)*C(K)
      1 -P**2*1152D0/(PI**6)*(-1D0)**F/(S**2)))/((R**2+(P*S)**2)**2)
16  CONTINUE
36  FORMAT(1HK,58H SUMREQ
      1SUMSEQ)
      WRITE(6,36)
      DO 34 I=1,L
      SUMR=0D0
      SUMS=0D0
      E=I-1
      K=I+L
      S=2*I-1
      DO 35 J=1,L
      F=J-1
      R=2*J-1
      SUMR =SUMR+(-1D0)**F*(R**2-(P*S)**2)*ARS(J,I)
35  SUMS=SUMS+(-1D0)**F*ARS(I,J)
      SUMR=-SUMR+C(K) -48D0/(PI**4)*P**2*(-1D0)**E/(S**2)
37  FORMAT(1H, 18X,2D24.16)
34  WRITE(6,37) SUMR,SUMS
      WRITE(6,13) P
13  FORMAT(1H3,3H P= F6.2,38H THE COEFFICIENTS ARS C(S). E.0)
      DO 14 I=1,5
26  FORMAT(1HK,5F12.6)
14  WRITE(6,26)(ARS(I,J),J=1,5)
C  CALCULATION OF THE X - VELOCITY COMPONENT
      WRITE(6,22) P
22  FORMAT(1H2,3H P= F6.2,23H X - VELOCITY )
      WRITE(6,28)
28  FORMAT(1HK, 70H Y=0 PI/20 2PI/20 3PI/20 4PI/20 5PI/20 6PI/20 7
      1PI/20 8PI/20 9PI/20)
      X=-PI/10D0

```



```

DO18 N=1,6
E=N-1
WRITE(6,29) E
29 FORMAT(1HK,2HX=I1,5HPI/10)
X=X+PI/10D0
Y=-PI/20D0
DO19 K=1,10
Y=Y+PI/20D0
US(K)=0D0
DO20 I=1,L
UR(K)=0D0
DO21 J=1,L
R=2*J-1
21 UR(K)=UR(K)+ARS(J,I)*SIN(R*X)
S=2*I-1
20 US(K)=US(K)+UR(K)*S*COS(S*Y)
UP(K)=6D0*P*X*((Y/PI)**2-.25D0)/PI
19 U(K)= PI/2D0*P*US(K)
WRITE(6,25)(U(K),K=1,10)
23 FORMAT(1HJ,10D13.5)
25 FORMAT(1HJ,10F7.3)
WRITE(6,30)
30 FORMAT(1HJ,40H PARABOLIC )
18 WRITE(6,25)(UP(K),K=1,10)
C CALCULATION OF THE Y - VELOCITY COMPONENT
WRITE(6,31)
31 FORMAT(1H2,35H Y - VELOCITY )
WRITE(6,82)
82 FORMAT(1HK, 70H X=0 PI/20 2PI/20 3PI/20 4PI/20 5PI/20 6PI/20 7
1PI/20 8PI/20 9PI/20)
Y=-PI/10D0
DO 78 N=1,6
E=N-1
WRITE(6,79) E
79 FORMAT(1HK,2HY=I1,5HPI/10)
Y=Y+PI/10D0
X=-PI/20D0
DO 80 K=1,10
X=X+PI/20D0
VR(K)=0D0
DO 70 I=1,L
VS(K)=0D0
DO 71 J=1,L
S=2*J-1
71 VS(K)=VS(K)+ARS(I,J)*SIN(S*Y)
R=2*I-1
70 VR(K)=VR(K)+VS(K)*R*COS(R*X)
VP(K)=6D0*Y*(.25D0-(Y/PI)**2/3D0)/PI
80 V(K)=-PI/2D0*VR(K)
WRITE(6,25)(V(K),K=1,10)
WRITE(6,30)
78 WRITE(6,25)(VP(K),K=1,10)
100 CONTINUE
7 STOP
END

```


P= 1.00 THE COEFFICIENTS ARE C(S). E.O

-0.012555	-0.010149	0.001627	-0.000449	0.000169
0.003400	0.002238	-0.000701	0.000248	-0.000104
-0.000866	-0.000415	0.000226	-0.000107	0.000053
0.000291	0.000101	-0.000077	0.000047	-0.000027
-0.000121	-0.000031	0.000030	-0.000021	0.000014

P= 1.00 X - VELOCITY

Y=0 $\pi/20$ $2\pi/20$ $3\pi/20$ $4\pi/20$ $5\pi/20$ $6\pi/20$ $7\pi/20$ $8\pi/20$ $9\pi/20$

X=0 $\pi/10$

0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000

PARABOLIC

-0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000

X=1 $\pi/10$

-0.010 -0.009 -0.007 -0.004 -0.001 0.003 0.006 0.007 0.006 0.004

PARABOLIC

-0.150 -0.148 -0.144 -0.136 -0.126 -0.112 -0.096 -0.076 -0.054 -0.028

X=2 $\pi/10$

-0.023 -0.021 -0.017 -0.010 -0.002 0.006 0.013 0.017 0.016 0.010

PARABOLIC

-0.300 -0.297 -0.288 -0.273 -0.252 -0.225 -0.192 -0.153 -0.108 -0.057

X=3 $\pi/10$

-0.041 -0.039 -0.031 -0.020 -0.006 0.009 0.022 0.031 0.031 0.021

PARABOLIC

-0.450 -0.445 -0.432 -0.409 -0.378 -0.337 -0.288 -0.229 -0.162 -0.085

X=4 $\pi/10$

-0.062 -0.058 -0.049 -0.033 -0.013 0.009 0.030 0.047 0.053 0.040

PARABOLIC

-0.600 -0.594 -0.576 -0.546 -0.504 -0.450 -0.384 -0.306 -0.216 -0.114

X=5 $\pi/10$

-0.074 -0.070 -0.059 -0.041 -0.020 0.007 0.032 0.056 0.066 0.060

PARABOLIC

-0.750 -0.742 -0.720 -0.682 -0.630 -0.562 -0.480 -0.382 -0.270 -0.142

Y - VELOCITY

X=0 $\pi/20$ $2\pi/20$ $3\pi/20$ $4\pi/20$ $5\pi/20$ $6\pi/20$ $7\pi/20$ $8\pi/20$ $9\pi/20$

Y=0 $\pi/10$

-0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000

PARABOLIC

0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000

Y=1 $\pi/10$

0.009 0.009 0.010 0.012 0.014 0.017 0.019 0.020 0.018 0.013

PARABOLIC

0.148 0.148 0.148 0.148 0.148 0.148 0.148 0.148 0.148 0.148

Y=2 $\pi/10$

0.012 0.013 0.015 0.018 0.022 0.026 0.030 0.033 0.031 0.022

PARABOLIC

0.284 0.284 0.284 0.284 0.284 0.284 0.284 0.284 0.284 0.284

Y=3 $\pi/10$

0.010 0.010 0.012 0.015 0.019 0.024 0.028 0.033 0.033 0.025

PARABOLIC

0.396 0.396 0.396 0.396 0.396 0.396 0.396 0.396 0.396 0.396

Y=4 $\pi/10$

0.003 0.004 0.004 0.006 0.007 0.010 0.013 0.016 0.019 0.017

PARABOLIC

0.472 0.472 0.472 0.472 0.472 0.472 0.472 0.472 0.472 0.472

Y=5 $\pi/10$

-0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000

PARABOLIC

0.500 0.500 0.500 0.500 0.500 0.500 0.500 0.500 0.500 0.500

P= 3.00 THE COEFFICIENTS ARE C(S). E.0

-0.001691 -0.001401 0.000199 -0.000053 0.000020

0.001395 0.001128 -0.000181 0.000050 -0.000019

-0.000977 -0.000753 0.000148 -0.000043 0.000017

0.000618 0.000444 -0.000110 0.000035 -0.000014

-0.000378 -0.000249 0.000078 -0.000028 0.000012

P= 3.00 X - VELOCITY

Y=0 PI/20 2PI/20 3PI/20 4PI/20 5PI/20 6PI/20 7PI/20 8PI/20 9PI/20

X=0PI/10

0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000

PARABOLIC

-0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000

X=1PI/10

0.000 0.000 0.000 0.000 0.000 0.000 -0.000 -0.000 -0.000 -0.000

PARABOLIC

-0.450 -0.445 -0.432 -0.409 -0.378 -0.337 -0.288 -0.229 -0.162 -0.085

X=2PI/10

-0.000 -0.000 -0.000 0.000 0.000 0.000 0.000 0.000 -0.000 -0.000

PARABOLIC

-0.900 -0.891 -0.864 -0.819 -0.756 -0.675 -0.576 -0.459 -0.324 -0.171

X=3PI/10

-0.003 -0.002 -0.002 -0.001 0.000 0.001 0.002 0.002 0.001 0.001

PARABOLIC

-1.350 -1.336 -1.296 -1.228 -1.134 -1.012 -0.864 -0.688 -0.486 -0.256

X=4PI/10

-0.024 -0.022 -0.018 -0.011 -0.002 0.007 0.014 0.017 0.016 0.010

PARABOLIC

-1.800 -1.782 -1.728 -1.638 -1.512 -1.350 -1.152 -0.918 -0.648 -0.342

X=5PI/10

-0.073 -0.069 -0.059 -0.041 -0.019 0.008 0.033 0.056 0.066 0.056

PARABOLIC

-2.250 -2.227 -2.160 -2.047 -1.890 -1.687 -1.440 -1.147 -0.810 -0.427

Y - VELOCITY

X=0 $\pi/20$ $2\pi/20$ $3\pi/20$ $4\pi/20$ $5\pi/20$ $6\pi/20$ $7\pi/20$ $8\pi/20$ $9\pi/20$

Y=0 $\pi/10$

-0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000

PARABOLIC

0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000

Y=1 $\pi/10$

-0.000 0.000 -0.000 0.000 -0.000 0.001 0.002 0.006 0.013 0.021

PARABOLIC

0.148 0.148 0.148 0.148 0.148 0.148 0.148 0.148 0.148 0.148

Y=2 $\pi/10$

-0.001 0.000 -0.001 0.001 -0.000 0.002 0.002 0.010 0.019 0.035

PARABOLIC

0.284 0.284 0.284 0.284 0.284 0.284 0.284 0.284 0.284 0.284

Y=3 $\pi/10$

-0.001 0.001 -0.001 0.001 -0.001 0.002 0.001 0.008 0.016 0.035

PARABOLIC

0.396 0.396 0.396 0.396 0.396 0.396 0.396 0.396 0.396 0.396

Y=4 $\pi/10$

-0.001 0.001 -0.001 0.001 -0.001 0.001 -0.000 0.004 0.005 0.019

PARABOLIC

0.472 0.472 0.472 0.472 0.472 0.472 0.472 0.472 0.472 0.472

Y=5 $\pi/10$

0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000

PARABOLIC

0.500 0.500 0.500 0.500 0.500 0.500 0.500 0.500 0.500 0.500

P= 5.00 THE COEFFICIENTS ARE C(S). E.0

-0.000619 -0.000513 0.000072 -0.000019 0.000007

0.000576 0.000474 -0.000070 0.000019 -0.000007

-0.000502 -0.000406 0.000065 -0.000018 0.000007

0.000412 0.000325 -0.000058 0.000017 -0.000006

-0.000323 -0.000246 0.000050 -0.000015 0.000006

P= 5.00 X - VELOCITY

Y=0 $\pi/20$ $2\pi/20$ $3\pi/20$ $4\pi/20$ $5\pi/20$ $6\pi/20$ $7\pi/20$ $8\pi/20$ $9\pi/20$

X=0 $\pi/10$

0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000

PARABOLIC

-0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000 -0.000

X=1 $\pi/10$

0.000 0.000 0.000 0.000 0.000 0.000 -0.000 -0.000 -0.000 -0.000

PARABOLIC

-0.750 -0.742 -0.720 -0.682 -0.630 -0.563 -0.480 -0.382 -0.270 -0.142

X=2 $\pi/10$

0.000 0.000 0.000 0.000 0.000 0.000 -0.000 -0.000 -0.000 -0.000

PARABOLIC

-1.500 -1.485 -1.440 -1.365 -1.260 -1.125 -0.960 -0.765 -0.540 -0.285

X=3 $\pi/10$

0.000 0.000 0.000 0.000 0.000 0.000 -0.000 -0.000 -0.000 -0.000

PARABOLIC

-2.250 -2.227 -2.160 -2.047 -1.890 -1.688 -1.440 -1.147 -0.810 -0.427

X=4 $\pi/10$

-0.006 -0.005 -0.004 -0.002 0.000 0.002 0.003 0.004 0.003 0.001

PARABOLIC

-3.000 -2.970 -2.880 -2.730 -2.520 -2.250 -1.920 -1.530 -1.080 -0.570

X=5 $\pi/10$

-0.070 -0.066 -0.056 -0.038 -0.017 0.009 0.033 0.054 0.061 0.050

PARABOLIC

-3.750 -3.712 -3.600 -3.412 -3.150 -2.812 -2.400 -1.912 -1.350 -0.712

Y - VELOCITY

X=0 $\pi/20$ $2\pi/20$ $3\pi/20$ $4\pi/20$ $5\pi/20$ $6\pi/20$ $7\pi/20$ $8\pi/20$ $9\pi/20$

Y=0 $\pi/10$

-0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000 -0.000 0.000

PARABOL IC

0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000

Y=1 $\pi/10$

-0.001 0.001 -0.001 0.001 -0.001 0.001 -0.001 0.002 0.003 0.019

PARABOL IC

0.148 0.148 0.148 0.148 0.148 0.148 0.148 0.148 0.148 0.148

Y=2 $\pi/10$

-0.001 0.001 -0.001 0.001 -0.001 0.002 -0.002 0.003 0.003 0.030

PARABOL IC

0.284 0.284 0.284 0.284 0.284 0.284 0.284 0.284 0.284 0.284

Y=3 $\pi/10$

-0.001 0.001 -0.001 0.001 -0.002 0.002 -0.002 0.003 0.001 0.029

PARABOL IC

0.396 0.396 0.396 0.396 0.396 0.396 0.396 0.396 0.396 0.396

Y=4 $\pi/10$

-0.001 0.001 -0.001 0.001 -0.001 0.001 -0.001 0.002 -0.001 0.014

PARABOL IC

0.472 0.472 0.472 0.472 0.472 0.472 0.472 0.472 0.472 0.472

Y=5 $\pi/10$

0.000 0.000 0.000 0.000 0.000 -0.000 -0.000 -0.000 0.000 0.000

PARABOL IC

0.500 0.500 0.500 0.500 0.500 0.500 0.500 0.500 0.500 0.500

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